Acta Crystallographica Section A
Foundations of Crystallography
ISSN 0108-7673

Received 17 March 2010
Accepted 4 June 2010

Diffusion-equation method for crystallographic figure of merits

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Global optimization methods play a significant role in crystallography, particularly in structure solution from powder diffraction data. This paper presents the mathematical foundations for a diffusion-equation-based optimization method. The diffusion equation is best known for describing how heat propagates in matter. However, it has also attracted considerable attention as the basis for global optimization of a multimodal function [Piela *et al.* (1989). *J. Phys. Chem.* **93**, 3339–3346]. The method relies heavily on available analytical solutions for the diffusion equation. Here it is shown that such solutions can be obtained for two important crystallographic figure-of-merit (FOM) functions that fully account for space-group symmetry and allow the diffusion-equation solution to vary depending on whether atomic coordinates are fixed or not. The resulting expression is computationally efficient, taking the same order of floating-point operations to evaluate as the starting FOM function measured in terms of the number of atoms in the asymmetric unit. This opens the possibility of implementing diffusion-equation methods for crystallographic global optimization algorithms such as structure determination from powder diffraction data.

1. Notation not discussed in detail in the paper

h, **k** are 3 × 1 Miller indices vectors, for instance

$$\mathbf{h} = \begin{pmatrix} h \\ k \\ l \end{pmatrix}.$$

$|\mathbf{h}|^2 = \mathbf{h}^T\mathbf{h} = h^2 + k^2 + l^2$ denotes the square modulus of a Miller indices vector.

$\mathbf{x}_n^T = (x_n\ y_n\ z_n)$ is the transpose of the position vector $\mathbf{x}_n$ for the nth atom in fractional or Cartesian coordinates as specified in the text.

N denotes the number of atoms in the unit cell.

$N_{\rm a}$ denotes the number of atoms in the asymmetric unit cell.

$\sum_{n \neq m}^{N}$ is shorthand for the summation over n from 1 to N not including $n = m$.

$f_n(\mathbf{h})$ is the form factor for the nth atom.

G denotes the elements of a space group generating the equivalent atoms in the unit cell.

g is a space-group element.

$\sum_{g \in G}$ is the summation where g runs over all the elements in G.

$\sum_{g_1,g_2}$ is shorthand for $\sum_{g_1,g_2 \in G} = \sum_{g_1 \in G} \sum_{g_2 \in G}$.

$t' = 4\pi^2 t$ (both t and t' are referred to as the deformation parameter).

2. Introduction

Figures of merit (FOMs) are used in crystallography to compare observed diffraction data with data calculated from a model and are also the global optimization metrics for structure solution. Two common FOM examples are the weighted sum of the squared differences between observed and calculated structure-factor amplitudes and the equivalent function for structure-factor intensities. The dimensionality of the surface of a crystallographic FOM is the number of degrees of freedom of the model. Even for systems with only a few degrees of freedom, the FOM surface is likely to contain a huge number of local minima and locating the deepest, global minimum is generally a major challenge. Many methods exist for solving the global optimization problem of locating the global minimum of a multi-minimum function. One such method is the diffusion-equation method (Piela *et al.*, 1989). Detailed descriptions and applications of this method are extensively discussed in the literature (*e.g.* Kostrowicki & Scheraga, 1992; Wawak *et al.*, 1996; Straub, 1998; Hart *et al.*, 2000). The surface of a common crystallographic FOM may be expected to consist of sharp features below an otherwise featureless landscape. The diffusion-equation method seeks to broaden these minima to a point where they merge into a surface that consists of one, or a few, minima. It proceeds by locating these minima and tracing them back to minima of the

doi:10.1107/S0108767310021380

unsmoothed surface by gradually reversing the smoothing process (note that the position of the minima may move as the surface becomes smoother). The presumption is that one of these minima will be the global minimum of the unsmoothed function.

The potential success of applying the diffusion-equation method is critically dependent on the computational cost of evaluating the smoothed function. In this work, we find analytical solutions to the diffusion equation for the intensity chi-squared FOM and a restraint FOM. For the intensity chi-squared FOM, we demonstrate how to take space-group symmetry fully into account and the effect on the diffusion-equation solution when the coordinates of the atoms are fixed, for example, on special positions or planes. Importantly, we derive an expression for the smoothed FOM that scales computationally of order $O(N_a)$ floating-point operations, which is the same as for the starting FOM.

3. Solving the diffusion equation

The diffusion-equation method can be expressed succinctly as the time reversal of solutions of the partial differential equation

$$\partial u(\mathbf{x}; t)/\partial t = \nabla^2 u(\mathbf{x}; t),$$

where the initial condition is $u(\mathbf{x}; t = 0) = \mathrm{FOM}(\mathbf{x})$. Minima of $u(\mathbf{x}; t)$ are located for a value of t where $u(\mathbf{x}; t)$ contains one, or a few, minima and these minima are traced back to $t = 0$. In this work, the coordinates of the vector $\mathbf{x}$ are the degrees of freedom of the FOM, which are atomic coordinates of atoms in the asymmetric unit.

4. The intensity χ^2 figure of merit

Here we find the solution to the diffusion equation for the intensity chi-squared FOM used for the analysis of both powder diffraction data and single-crystal data. We define

$$\chi^2 = \sum_{\mathbf{h}} \sum_{\mathbf{k}} (I_{\mathbf{h}}^{\mathrm{obs}} - sI_{\mathbf{h}}) w_{\mathbf{hk}} (I_{\mathbf{k}}^{\mathrm{obs}} - sI_{\mathbf{k}}), \tag{1}$$

where $\mathbf{h}$ and $\mathbf{k}$ are Miller indices, s is a scale factor, and $I_{\mathbf{h}}^{\mathrm{obs}}$ and $I_{\mathbf{h}}$ are the observed and the calculated intensities, respectively, for the reflection with Miller indices $\mathbf{h}$. The weight $w_{\mathbf{hk}}$ describes the correlation between reflection $\mathbf{h}$ and $\mathbf{k}$. For single-crystal diffraction data all off-diagonal weight elements are zero and the diagonal elements equal $w_{\mathbf{hh}} = \sigma_{\mathbf{h}}^{-2}$, where $\sigma_{\mathbf{h}}$ represents an estimated standard deviation for the measured intensity $I_{\mathbf{h}}^{\mathrm{obs}}$. For powder diffraction data, the off-diagonal elements of the weight matrix, $w_{\mathbf{hk}}$, are in general non-zero because of reflection overlap.

Further, we write the reflection intensity as

$$I_{\mathbf{h}} = |F_{\mathbf{h}}|^2 = \sum_{m,n=1}^{N_a} f_m(\mathbf{h}) f_n(\mathbf{h}) \sum_{g_1, g_2 \in G} \exp\big(2\pi i \mathbf{h}^T (\mathbf{S}_{g_1}\mathbf{x}_m - \mathbf{S}_{g_2}\mathbf{x}_n)\big), \tag{2}$$

where N_a is the number of atoms in the asymmetric unit and $f_n(\mathbf{h})$ is the form factor for the nth atom. G is the space group of the crystal and g is a member of this group that maps the position $\mathbf{x}_n$ onto the position $\mathbf{S}_g\mathbf{x}_n = \mathbf{R}_g\mathbf{x}_n + \mathbf{t}_g$, where $\mathbf{R}_g$ and $\mathbf{t}_g$ are the rotational and translational components, respectively, of this symmetry operator.

We seek a diffusion-equation solution with the initial condition $u(\mathbf{x}; t = 0)$ equal to the FOM in equation (1). The coordinates of the atoms in the asymmetric unit are initially taken to be fractional coordinates. At one level, this task is straightforward since the diffusion equation is linear and the FOM in equation (1) consists of a sum of terms that all have a form that is a constant multiplied by a phase factor of the type

$$\xi = \exp\big(2\pi i (\mathbf{h}_1^T \mathbf{S}_{g_1} + \ldots + \mathbf{h}_P^T \mathbf{S}_{g_P}) \mathbf{x}_m\big). \tag{3}$$

Knowing that the diffusion-equation solution to $\exp(2\pi i \mathbf{hx})$ equals $\exp(-t'|\mathbf{h}|^2)\exp(2\pi i \mathbf{hx})$, it is seen that the diffusion-equation solution to equation (3) is

$$u(\mathbf{x}_m; t) = \exp(-t'|\mathbf{h}_1^T \mathbf{R}_{g_1} + \ldots + \mathbf{h}_P^T \mathbf{R}_{g_P}|^2)\xi.$$

The diffusion-equation solution is essentially a phase factor multiplied by a Debye–Waller like penalty factor that does not depend on the translation symmetry of the space-group operators. At a more detailed level, we see for instance from equation (2) that the expression of these Debye-–Waller like factors will depend on whether the atomic indices m and n in the double summation in that equation are equal or not equal. The full diffusion-equation solution to $I_{\mathbf{h}}$ may be written as

$$\begin{aligned} I_{\mathbf{h}}(t) = {} & \exp\big(-2t'|\mathbf{h}^T|^2\big) \sum_{m \neq n}^{N_a} f_m(\mathbf{h}) f_n(\mathbf{h}) \\ & \times \sum_{g_1, g_2 \in G} \exp\big(2\pi i \mathbf{h}^T (\mathbf{S}_{g_1}\mathbf{x}_m - \mathbf{S}_{g_2}\mathbf{x}_n)\big) \\ & + \sum_{n=1}^{N_a} [f_n(\mathbf{h})]^2 \sum_{g_1, g_2 \in G} \big[\exp\big(-t'|\mathbf{h}^T(\mathbf{R}_{g_1} - \mathbf{R}_{g_2})|^2\big) \\ & \times \exp\big(2\pi i \mathbf{h}^T (\mathbf{S}_{g_1} - \mathbf{S}_{g_2})\mathbf{x}_n\big)\big]. \end{aligned} \tag{4}$$

As written in equation (4) $I_{\mathbf{h}}(t)$ takes of the order $O(N_a^2)$ floating-point operations to calculate whereas $I_{\mathbf{h}}$ in equation (2) takes only of the order $O(N_a)$ floating-point operations. In order to write equation (4), and eventually equation (1), in a form that takes $O(N_a)$ operations rather than $O(N_a^2)$ operations, it is useful to introduce the following notation:

$$\begin{aligned} & \Phi_m(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_P; t) \\ & \quad = f_m(\mathbf{h}_1) f_m(\mathbf{h}_2) \ldots f_m(\mathbf{h}_p) \\ & \quad \times \sum_{g_1, g_2, \ldots, g_P} \exp\big(2\pi i (\mathbf{h}_1^T \mathbf{S}_{g_1} + \ldots + \mathbf{h}_P^T \mathbf{S}_{g_P})\mathbf{x}_m \\ & \quad - t'|\mathbf{h}_1^T \mathbf{R}_{g_1} + \ldots + \mathbf{h}_P^T \mathbf{R}_{g_P}|^2\big). \end{aligned} \tag{5}$$

Apart from the Debye–Waller like penalty factor, Φ_m is recognized (see, for example, Giacovazzo, 1998) to be a product of trigonometric structure factors.

In a similar way to how a structure factor is written as a sum of trigonometric structure factors, introduce

$$F_{[\mathbf{h}_{11},\ldots,\mathbf{h}_{1P_1}][\mathbf{h}_{21},\ldots,\mathbf{h}_{2P_2}]\ldots[\mathbf{h}_{K1},\ldots,\mathbf{h}_{KP_K}]} = \sum_{m=1}^{N_{\rm a}} \prod_{i=1}^{K} \Phi_m(\mathbf{h}_{i1}, \ldots, \mathbf{h}_{iP_i}; t), \tag{6}$$

where $\mathbf{h}_{11}, \ldots, \mathbf{h}_{1P_1}$, $\mathbf{h}_{21}, \ldots, \mathbf{h}_{2P_2}$ and $\mathbf{h}_{K1}, \ldots, \mathbf{h}_{KP_K}$ are K Miller index vector sets of sizes $P_1, P_2, \ldots, P_K$, respectively. The reason for the square-bracket notation in equation (6) is to distinguish between say $F_{[\mathbf{h}_1][\mathbf{h}_2]}$ and $F_{[\mathbf{h}_1,\mathbf{h}_2]}$, which are identical apart from different t-dependent Debye–Waller like penalty terms.

Using equation (5) the equation in (4) can be written as

$$I_{\mathbf{h}}(t) = \sum_{m=1}^{N_{\rm a}} \sum_{n\neq m}^{N_{\rm a}} \Phi_m(\mathbf{h}; t)\Phi_n(-\mathbf{h}; t) + \sum_{m=1}^{N_{\rm a}} \Phi_m(\mathbf{h}, -\mathbf{h}; t). \tag{7}$$

The double summation in equation (7) takes of the order $O(N_{\rm a}^2)$ floating-point operations to calculate, but rewriting this term as

$$\sum_{m=1}^{N_{\rm a}} \sum_{n\neq m}^{N_{\rm a}} \Phi_m(\mathbf{h}; t)\Phi_n(-\mathbf{h}; t) = \sum_{m=1}^{N_{\rm a}} \Phi_m(\mathbf{h}; t) \sum_{n=1}^{N_{\rm a}} \Phi_n(-\mathbf{h}; t) - \sum_{m=1}^{N_{\rm a}} \Phi_m(\mathbf{h}; t)\Phi_m(-\mathbf{h}; t) \tag{8}$$

takes of the order $O(N_{\rm a})$. Using equation (8) and the notation in equation (6), we see that the diffusion-equation solution to $I_{\mathbf{h}}$ can be written as

$$I_{\mathbf{h}}(t) = |F_{[\mathbf{h}]}|^2 + F_{[\mathbf{h},-\mathbf{h}]} - F_{[\mathbf{h}][-\mathbf{h}]}.$$

Finding the diffusion-equation solution to equation (1) also requires finding the solution to

$$\begin{aligned} I_{\mathbf{h}} I_{\mathbf{k}} = {} & \sum_{m,n,p,q=1}^{N_{\rm a}} f_m(\mathbf{h}) f_n(\mathbf{h}) f_p(\mathbf{k}) f_q(\mathbf{k}) \\ & \times \sum_{g_1,g_2,g_3,g_4} \exp\big(2\pi i[\mathbf{h}^T(\mathbf{S}_{g_1}\mathbf{x}_m - \mathbf{S}_{g_2}\mathbf{x}_n) \\ & + \mathbf{k}^T(\mathbf{S}_{g_3}\mathbf{x}_p - \mathbf{S}_{g_4}\mathbf{x}_q)]\big). \end{aligned} \tag{9}$$

Whether the atomic indices m, n, p and q in equation (9) are different or equal gives rise to different Debye–Waller like penalty terms, as was the case for $I_{\mathbf{h}}$ in equation (7). Denoting $I_{\mathbf{hk}} \equiv I_{\mathbf{h}} I_{\mathbf{k}}$, then equation (10) lists the diffusion-equation solution to $I_{\mathbf{hk}}$:

$$\begin{aligned} I_{\mathbf{hk}}(t) = {} & \sum_{m=n=p=q} \Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}, -\mathbf{k}; t) \\ & + \sum_{m=n=p, m\neq q} \Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}; t)\Phi_q(-\mathbf{k}; t) \\ & + \sum_{m=n=q, m\neq p} \Phi_m(\mathbf{h}, -\mathbf{h}, -\mathbf{k}; t)\Phi_p(\mathbf{k}; t) \\ & + \sum_{m=n, p=q, m\neq p} \Phi_m(\mathbf{h}, -\mathbf{h}; t)\Phi_p(\mathbf{k}, -\mathbf{k}; t) \\ & + \sum_{m=n, m\neq p, m\neq q, p\neq q} \Phi_m(\mathbf{h}, -\mathbf{h}; t)\Phi_p(\mathbf{k}; t)\Phi_q(-\mathbf{k}; t) \\ & + \sum_{n=p=q, m\neq n} \Phi_n(-\mathbf{h}, \mathbf{k}, -\mathbf{k}; t)\Phi_m(\mathbf{h}; t) \\ & + \sum_{m=p=q, m\neq n} \Phi_m(\mathbf{h}, \mathbf{k}, -\mathbf{k}; t)\Phi_n(-\mathbf{h}; t) \\ & + \sum_{p=q, m\neq n, m\neq p, n\neq p} \Phi_p(\mathbf{k}, -\mathbf{k}; t)\Phi_m(\mathbf{h}; t)\Phi_n(-\mathbf{h}; t) \\ & + \sum_{m=p, n=q, m\neq n} \Phi_m(\mathbf{h}, \mathbf{k}; t)\Phi_n(-\mathbf{h}, -\mathbf{k}; t) \\ & + \sum_{m=q, n=p, m\neq n} \Phi_m(\mathbf{h}, -\mathbf{k}; t)\Phi_n(-\mathbf{h}, \mathbf{k}; t) \\ & + \sum_{m=p, m\neq n, m\neq q, n\neq q} \Phi_m(\mathbf{h}, \mathbf{k}; t)\Phi_n(-\mathbf{h}; t)\Phi_q(-\mathbf{k}; t) \\ & + \sum_{n=q, m\neq n, m\neq p, n\neq p} \Phi_n(-\mathbf{h}, -\mathbf{k}; t)\Phi_m(\mathbf{h}; t)\Phi_p(\mathbf{k}; t) \\ & + \sum_{m=q, m\neq n, m\neq p, n\neq p} \Phi_m(\mathbf{h}, -\mathbf{k}; t)\Phi_n(-\mathbf{h}; t)\Phi_p(\mathbf{k}; t) \\ & + \sum_{n=p, m\neq n, m\neq q, n\neq q} \Phi_n(-\mathbf{h}, \mathbf{k}; t)\Phi_m(\mathbf{h}; t)\Phi_q(-\mathbf{k}; t) \\ & + \sum_{m,n,p,q \text{ all different}} \Phi_m(\mathbf{h}; t)\Phi_n(-\mathbf{h}; t)\Phi_p(\mathbf{k}; t)\Phi_q(-\mathbf{k}; t). \end{aligned} \tag{10}$$

It costs of the order $O(N_{\rm a}^4)$ to evaluate the last term in equation (10). Each of the terms in equation (10) can be rewritten such that the computational cost of calculating it takes of the order $O(N_{\rm a})$ to compute using expressions similar to equation (8). For example, for the second, third, fourth, sixth, seventh, ninth and tenth terms, the expression in equation (8) is directly applicable. For a triple summation term such as the second from last term in equation (10), we can use

$$\begin{aligned} & \sum_{n=p, m\neq n, m\neq q, n\neq q} \Phi_n(-\mathbf{h}, \mathbf{k}; t)\Phi_m(\mathbf{h}; t)\Phi_q(-\mathbf{k}; t) \\ & = F_{[\mathbf{h}]}F_{[-\mathbf{h},\mathbf{k}]}F_{[-\mathbf{k}]} - F_{[\mathbf{h}][-\mathbf{h},\mathbf{k}]}F_{[-\mathbf{k}]} \\ & \quad - F_{[\mathbf{h}][-\mathbf{k}]}F_{[-\mathbf{h},\mathbf{k}]} - F_{[\mathbf{h}]}F_{[-\mathbf{h},\mathbf{k}][-\mathbf{k}]} + 2F_{[\mathbf{h}][-\mathbf{h},\mathbf{k}][-\mathbf{k}]}. \end{aligned}$$

Carrying this analysis through for all terms in equation (10) and inserting $I_{\mathbf{hk}}(t)$ and $I_{\mathbf{h}}(t)$ into the FOM function in equation (1), we eventually find the diffusion-equation solution to equation (1), which for example can be grouped into the following formula:

$$\begin{aligned} \chi^2(t) = s^2 \sum_{\mathbf{h},\mathbf{k}} w_{\mathbf{hk}} \big\{ & F^{\rm iv}(\mathbf{h}, \mathbf{k}; t) \\ & + \big(F_{[\mathbf{k}]}F^{\rm iii}(\mathbf{h}, -\mathbf{k}; t) + \text{c.c.}\big) + \big(F_{[\mathbf{h}]}F^{\rm iii}(\mathbf{k}, -\mathbf{h}; t) + \text{c.c.}\big) \\ & + |F_{[\mathbf{h}]}F_{[\mathbf{k}]} + F^{\rm ii}(\mathbf{h}, \mathbf{k}; t)|^2 + |F_{[\mathbf{h}]}F_{[-\mathbf{k}]} + F^{\rm ii}(\mathbf{h}, -\mathbf{k}; t)|^2 \\ & - 2|F_{[\mathbf{h}]}|^2|F_{[\mathbf{k}]}|^2 + \big(I_{\mathbf{h}}(t) - I_{\mathbf{h}}^{\rm obs}/s\big)\big(I_{\mathbf{k}}(t) - I_{\mathbf{k}}^{\rm obs}/s\big)\big\}, \end{aligned} \tag{11}$$

where c.c. stands for complex conjugate, $F^{\rm ii}(\mathbf{h}, \mathbf{k}; t) = F_{[\mathbf{h},\mathbf{k}]} - F_{[\mathbf{h}][\mathbf{k}]}$, $F^{\rm iii}(\mathbf{h}, \mathbf{k}; t) = F_{[\mathbf{h},-\mathbf{h},\mathbf{k}]} - F_{[\mathbf{h},\mathbf{k}][-\mathbf{h}]} - F_{[-\mathbf{h},\mathbf{k}][\mathbf{h}]} - F_{[\mathbf{h},-\mathbf{h}][\mathbf{k}]} + 2F_{[\mathbf{h}][-\mathbf{h}][\mathbf{k}]}$ and $F^{\rm iv}(\mathbf{h}, \mathbf{k}; t) = F_{[\mathbf{h},-\mathbf{h},\mathbf{k},-\mathbf{k}]} - 6F_{[\mathbf{h}][-\mathbf{h}][\mathbf{k}][-\mathbf{k}]} - F_{[\mathbf{h},-\mathbf{h},-\mathbf{k}][\mathbf{k}]} + 2F_{[\mathbf{h},-\mathbf{k}][-\mathbf{h}][\mathbf{k}]} + 2F_{[-\mathbf{h},-\mathbf{k}][\mathbf{h}][\mathbf{k}]} - F_{[\mathbf{h},-\mathbf{h},\mathbf{k}][-\mathbf{k}]} + 2F_{[-\mathbf{h},\mathbf{k}][\mathbf{h}][-\mathbf{k}]} + 2F_{[\mathbf{h},\mathbf{k}][-\mathbf{h}][-\mathbf{k}]} - F_{[\mathbf{h}][-\mathbf{h},\mathbf{k},-\mathbf{k}]} + 2F_{[\mathbf{h},-\mathbf{h}][\mathbf{k}][-\mathbf{k}]} + 2F_{[\mathbf{h}][-\mathbf{h}][\mathbf{k},-\mathbf{k}]} - F_{[\mathbf{h},\mathbf{k},-\mathbf{k}][-\mathbf{h}]} - F_{[\mathbf{h},-\mathbf{h}][\mathbf{k},-\mathbf{k}]} - F_{[\mathbf{h},\mathbf{k}][-\mathbf{h},-\mathbf{k}]} - F_{[\mathbf{h},-\mathbf{k}][-\mathbf{h},\mathbf{k}]}$. This formula takes of the order $O(N_{\rm a})$ to evaluate.

4.1. The case of fixed atoms and coordinate transformations

The atom position vector $\mathbf{x}_m$ and space-group operator S_g are expressed with respect to a crystal coordinate system in equation (5). Using a notation similar to that found in Giacovazzo *et al.* (2002),

$$\mathbf{x}'_m = \mathbf{M}_m \mathbf{x}_m, \tag{12}$$

equation (12) relates the coordinates of atom n between two different coordinate systems as defined by the coordinate transformation matrix $\mathbf{M}_m$. If, for example, the columns of $\mathbf{M}_m$ equal the lattice vectors of the unit cell, then equation (12) represents the transformation from fractional to Cartesian coordinates. The gradient of the phase term in equation (3) with respect to $\mathbf{x}'_m$, rather than $\mathbf{x}_m$, is

$$\nabla'\xi = \nabla\xi\mathbf{M}_m^{-1} = 2\pi i(\mathbf{h}_1^T\mathbf{S}_{g_1} + \ldots + \mathbf{h}_P^T\mathbf{S}_{g_P})\mathbf{M}_m^{-1}\xi. \tag{13}$$

If $\mathbf{M}_m$ represents the transformation from fractional to Cartesian coordinates, the rows of $\mathbf{M}_m^{-1}$ equal the reciprocal-lattice vectors.

Using equation (13) it is seen that the diffusion-equation solution with respect to $\mathbf{x}'_m$ is obtained by substituting equation (5) with

$$\begin{aligned}\Phi_m(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_P; t') &= f_m(\mathbf{h}_1) f_m(\mathbf{h}_2) \ldots f_m(\mathbf{h}_p) \\ &\times \sum_{g_1, g_2, \ldots, g_P} \exp\big(2\pi i(\mathbf{h}_1^T\mathbf{S}_{g_1} + \ldots + \mathbf{h}_P^T\mathbf{S}_{g_P})\mathbf{x}_m \\ &- t'|[\mathbf{h}_1^T\mathbf{R}_{g_1} + \ldots + \mathbf{h}_P^T\mathbf{R}_{g_P}]\mathbf{M}_m^{-1}|^2\big).\end{aligned} \tag{14}$$

It is perhaps surprisingly easy to incorporate into the diffusion-equation solution in equation (11) the concept of atomic coordinates that are either fixed or not fixed. Simply substitute each atom $\mathbf{x}_m$ by $(\mathbf{P}_m + \overline{\mathbf{P}}_m)\mathbf{x}_m$, where $\mathbf{P}_m$ and $\overline{\mathbf{P}}_m$ are both diagonal matrices with diagonal elements that are either zero or one and their sum is the identity matrix. $\mathbf{P}_m\mathbf{x}_m$ is interpreted as the component of $\mathbf{x}_m$ that is not fixed (allowed to vary) and $\overline{\mathbf{P}}_m\mathbf{x}_m$ is the component which stays constant (is fixed). Since $\overline{\mathbf{P}}_m\mathbf{x}_m$ is treated as a constant, this component will not contribute to the Debye–Waller like penalty factor and the diffusion-equation solution taking into account fixed coordinates is obtained by substituting equation (14) with

$$\begin{aligned}\Phi_m(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_P; t') &= f_m(\mathbf{h}_1) f_m(\mathbf{h}_2) \ldots f_m(\mathbf{h}_p) \\ &\times \sum_{g_1, g_2, \ldots, g_P} \exp\big(2\pi i(\mathbf{h}_1^T\mathbf{S}_{g_1} + \ldots + \mathbf{h}_P^T\mathbf{S}_{g_P})\mathbf{x}_m \\ &- t'|[\mathbf{h}_1^T\mathbf{R}_{g_1} + \ldots + \mathbf{h}_P^T\mathbf{R}_{g_P}]\mathbf{M}_m^{-1}\mathbf{P}_m|^2\big).\end{aligned}$$

The matrices $\mathbf{P}_m$ and $\overline{\mathbf{P}}_m$ for an atom can be used to fix the mth atom to a point or an arbitrary line or plane when combined with suitable transformation matrices $\mathbf{M}_m^{-1}$. Consider, for example, the case of an atom fixed to move along the direction [110] relative to a crystal coordinate system. Choose for the mth atom

$$\mathbf{M}_m = \begin{pmatrix} 2^{-1/2} & -2^{-1/2} & 0 \\ 2^{-1/2} & 2^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This corresponds to rotating the crystal counterclockwise by 45° or rotation of the coordinate system by 45° clockwise resulting in the y axis now being positioned along the [110] direction in the old coordinate system.

$$\mathbf{P}_m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then has the effect that the diffusion-equation solution only includes a diffusion penalty contribution for the mth atom along the direction [110] measured in the crystal coordinate system.

4.2. Calculating $\chi^2(t)$

The time it takes to calculate equation (11) is proportional to the number of atoms in the asymmetric unit multiplied by the number of data points. The choice of space group and the type of constraints applied (projection matrices) in general affect the evaluation time of equation (11). Here a procedure for calculating equation (11) is discussed.

For example, we need to calculate $F_{[\mathbf{h}]} = \sum_m \Phi_m(\mathbf{h}; t)$, where

$$\Phi_m(\mathbf{h}; t) = \sum_g \exp(2\pi i\mathbf{h}^T\mathbf{S}_g\mathbf{x}_m)\big[f_m(\mathbf{h})\exp(-t'|\mathbf{h}^T\mathbf{R}_g\mathbf{M}_m^{-1}\mathbf{P}_m|^2)\big]. \tag{15}$$

An approach is to first calculate for each reflection $\mathbf{h}$ the squared lengths $|\mathbf{h}^T\mathbf{R}_g\mathbf{M}_m^{-1}\mathbf{P}_m|^2$ for all m. This only needs to be performed once for a given data set. For a given value of the t parameter, the factors in the square bracket in equation (15) are what remain to recalculate the phases for each new configuration of the atoms. Equation (11) requires, in addition to $\Phi_m(\mathbf{h}; t)$, the evaluation of $\Phi_m(\mathbf{h}, \mathbf{k}; t)$, $\Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}; t)$ and $\Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}, -\mathbf{k}; t)$. $\Phi_m(\mathbf{h}, \mathbf{k}; t)$ reads

$$\begin{aligned}\Phi_m&(\mathbf{h}, \mathbf{k}; t) \\ &= \sum_{g_1, g_2} \big\{\exp\big(2\pi i(\mathbf{h}^T\mathbf{S}_{g_1} + \mathbf{k}^T\mathbf{S}_{g_2})\mathbf{x}_m\big) \\ &\quad \times \big[f_m(\mathbf{h}) f_m(\mathbf{k}) \exp(-t'|[\mathbf{h}^T\mathbf{R}_{g_1} + \mathbf{k}^T\mathbf{R}_{g_2}]\mathbf{M}_m^{-1}\mathbf{P}_m|^2)\big]\big\}.\end{aligned} \tag{16}$$

Once a set of phases, $\exp(2\pi i\mathbf{h}^T\mathbf{S}_g\mathbf{x}_m)$, has been calculated, evaluating $\exp(2\pi i(\mathbf{h}^T\mathbf{S}_g + \mathbf{k}^T\mathbf{S}_{g'})\mathbf{x}_m)$ takes at most one complex multiplication. A two-dimensional array of time-dependent factors, in the square brackets in equation (16), needs to be evaluated; again, this only needs to be done once for a given value of the deformation parameter. Similar considerations can be made for the calculation of the trigonometric structure factors $\Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}; t)$ and $\Phi_m(\mathbf{h}, -\mathbf{h}, \mathbf{k}, -\mathbf{k}; t)$, where the latter requires a four-dimensional array of time-dependent factors. Once these have been calculated, evaluating $F^{\rm ii}$, $F^{\rm iii}$, $F^{\rm iv}$ and $\chi^2(t)$ in equation (11) is straightforward.

4.3. The case of $P1$ symmetry and single-crystal data

For the special case of space group $P1$ and all off-diagonal weight matrix elements equal to zero, the diffusion-equation solution in equation (11), allowing atomic coordinates to be fixed, reduces to

$$\chi^2(t) = s^2 \sum_{\mathbf{h}} w_{\mathbf{hh}} \big\{ F^{\rm iv}(\mathbf{h}, \mathbf{h}; t) + 2\big(F_{[\mathbf{h}]} F^{\rm iii}(\mathbf{h}, -\mathbf{h}; t) + {\rm c.c.}\big) + |(F_{[\mathbf{h}]})^2 + F^{\rm ii}(\mathbf{h}, \mathbf{h}; t)|^2 + [I_{\mathbf{h}}(t)]^2 - 2|F_{[\mathbf{h}]}|^4 + \big(I_{\mathbf{h}}(t) - I_{\mathbf{h}}^{\rm obs}/s\big)^2 \big\}, \tag{17}$$

where c.c. stands for complex conjugate and

$$\begin{aligned} \beta_m &= \exp(-t'|\mathbf{h}^T \mathbf{M}_m^{-1} \mathbf{P}_m|^2), \\ F^{\rm iv}(\mathbf{h}, \mathbf{h}; t) &= -\sum_{m=1}^N f_m^4 (1 - \beta_m^2)^4, \\ F^{\rm iii}(\mathbf{h}, -\mathbf{h}; t) &= -\sum_{m=1}^N f_m^3 \exp(-2\pi i \mathbf{h}^T \mathbf{x}_m) \beta_m (1 - \beta_m^2)^2, \\ F^{\rm ii}(\mathbf{h}, -\mathbf{h}; t) &= \sum_{m=1}^N f_m^2 (1 - \beta_m^2), \\ F^{\rm ii}(\mathbf{h}, \mathbf{h}; t) &= \sum_{m=1}^N f_m^2 \exp(4\pi i \mathbf{h}^T \mathbf{x}_m) \beta_m^2 (\beta_m^2 - 1), \\ F_{[\mathbf{h}]} &= \sum_{m=1}^N f_m \exp(2\pi i \mathbf{h}^T \mathbf{x}_m) \beta_m, \\ I_{\mathbf{h}}(t) &= |F_{[\mathbf{h}]}|^2 + F^{\rm ii}(\mathbf{h}, -\mathbf{h}; t). \end{aligned}$$

It is worth considering the nature of the $P1$ single-crystal χ^2 function at the two limits of the diffusion process when $t \to 0$ and $t \to \infty$, and where the constraint matrices are all identical and equal to the identity matrix, implying that $\beta_m = \beta$ for all the atoms. For $\beta = 1$, that is for $t = 0$, equation (17) reduces to the conventional single-crystal χ^2 function given by

$$\chi^2(t = 0) = s^2 \sum_{\mathbf{h}} w_{\mathbf{hh}} (I_{\mathbf{h}} - I_{\mathbf{h}}^{\rm obs}/s)^2.$$

For $\beta = 0$, *i.e.* the extreme condition of the diffusion equation corresponding to an infinite time, $t \to \infty$, equation (17) reduces to

$$\chi^2(t = \infty) = s^2 \sum_{\mathbf{h}} w_{\mathbf{hh}} \left\{ \left(\sum_{m=1}^N f_m^2 \right)^2 - \sum_{m=1}^N f_m^4 + \left(\sum_{m=1}^N f_m^2 - I_{\mathbf{h}}^{\rm obs}/s \right)^2 \right\}.$$

The most noticeable term in this equation is $(\sum_{m=1}^N f_m^2 - I_{\mathbf{h}}^{\rm obs}/s)^2$. The intensities $I_{\mathbf{h}}^{\rm obs}$ are the Fourier components of a Patterson function whose maxima are located at the interatomic vector positions, $(\mathbf{x}_m - \mathbf{x}_{m'})$, and the term $\sum_{m=1}^N f_m^2$ is the Fourier transform of a constant featureless Patterson function. The diffusion equation has effectively rendered the Patterson function featureless and the analytical time reversal of the diffusion method smoothly recreates the Patterson function. The nature of the first non-constant addition to the Patterson function as t is reduced from infinity is found by first isolating the non-constant terms in equation (17) to lowest order in β:

$$\chi^2(t = {\rm large}) = \chi^2(t = \infty) + 2s^2 \sum_{\mathbf{h}} w_{\mathbf{hh}} \beta^2 \times \sum_{m=1}^N \sum_{n \neq m}^N f_m f_n \left[2 \sum_{p=1}^N f_p^2 - f_m^2 - f_n^2 - I_{\mathbf{h}}^{\rm obs}/s \right] \times \cos\big(2\pi i \mathbf{h}^T (\mathbf{x}_m - \mathbf{x}_n)\big).$$

For small β, only the longest d-spacing reflections will contribute and thus the first feature that appears to spontaneously break the featureless symmetry of the $t = \infty$ Patterson map is associated with the first reflection. In the other limit, as t is increased away from zero, the first blurring of equation (17) scales, in many places, as $(1 - \beta^2)$ and can be considered to be closely similar to an additional Debye–Waller term. Thus the analytical time reversal of equation (17) is similar to the least-squares fitting of a Patterson function that is initially featureless, develops structure associated with the lowest-resolution reflections and finally sharpens into the detailed Patterson function that is associated with the observed diffraction data.

5. Restraints

A commonly applied restraint in crystallography is the bond-length restraint, which favours certain distances between two atoms over others. Consider the following restraint function used by Kostrowicki & Scheraga (1992) in an application of the diffusion-equation method:

$$f(\mathbf{x}) = w_{nm} (|\mathbf{x}_n - \mathbf{x}_m|^2 - x_{nm}^2)^2, \tag{18}$$

where $\mathbf{x}$ denotes the collective Cartesian coordinates of the mth and nth atoms, and $|\mathbf{x}_n - \mathbf{x}_m|$ equals the distance between atoms n and m. This function restrains the distance between these two atoms with a weight parameter given by w_{nm} and preferred distance x_{nm}. The diffusion-equation solution with $u(\mathbf{x}; t = 0) = f(\mathbf{x})$ is (Kostrowicki & Scheraga, 1992)

$$u(\mathbf{x}; t) = w_{nm} \{ [|\mathbf{x}_n - \mathbf{x}_m|^2 - (x_{nm}^2 - 20t)]^2 + 16 x_{nm}^2 t - 160 t^2 \}. \tag{19}$$

The expression in equation (19) can readily be extended to the scenario where one or both of the two atoms in equation (18) are fixed at a point, to a line or a plane. This is achieved using the matrices introduced in §4.1 to fix coordinates and the resulting diffusion-equation solution is

$$u(\mathbf{x}; t) = w_{nm} \Big(\{ |\mathbf{x}_n - \mathbf{x}_m|^2 - [x_{nm}^2 - 2t {\rm Tr}(\mathbf{P}_{nm})] \}^2 + 8t (\mathbf{x}_n - \mathbf{x}_m)^T \mathbf{P}_{nm} (\mathbf{x}_n - \mathbf{x}_m) + 20 t^2 {\rm Tr}(\mathbf{P}_{nm}) + 16 t^2 {\rm Tr}(\mathbf{P}_n \mathbf{P}_m) - 12 t^2 \sum_{i=1}^3 [(\mathbf{P}_n)_{ii}^2 + (\mathbf{P}_m)_{ii}^2] \Big), \tag{20}$$

where $\mathbf{P}_{nm} = \mathbf{P}_n + \mathbf{P}_m$, $(\mathbf{P}_n)_{ii}$ is the ith diagonal element of $\mathbf{P}_n$ and ${\rm Tr}(\mathbf{P}_{nm})$ is the trace of the matrix $\mathbf{P}_{nm}$. Notice that the last three terms in equation (20) do not depend on $\mathbf{x}$.

6. Conclusions

Analytical solutions to the diffusion-equation method for FOMs of interest to crystallography were derived which fully account for space-group symmetry and constraint information for the first time. This provides the framework for applying the diffusion-equation method to FOMs relevant to crystallography using information available about symmetry and constraints.

We thank Professor Schwarzenbach and the referees for their constructive and helpful comments.

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